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The Semilinear Second-Order Hyperbolic Equation with Data Strongly Singular at One Point

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Abstract. In this paper we study the initial value problem for the scalar semilinear strictly hyperbolic equation in multidimensional space with data strongly singular at one point. Under the assumption of the initial data being conormal with respect to one point and bounded or regular with a certain low degree, the existence of the solution to this problem is obtained; meanwhile, it is proved that the singularity of the solution will spread on the forward characteristic cone of the hyperbolic operator issuing from this point, and the solution is bounded and conormal with respect to this cone.

1. Introduction

The Cauchy problem of nonlinear hyperbolic equations with data weakly singular at one point has been studied by M. Beals [1] for the semilinear wave equation. Later on, J. M. Bony [5] and J. Y. Chemin [7]–[9] have also studied singularities of solutions to this kind of Cauchy problems for general nonlinear hyperbolic equations after establishing the theory of the second microlocalization (see [4]). Here we wish to study the conormal property of solutions to general second order hyperbolic equations with data being strongly singular at one point. The special case of the semilinear wave equation has been investigated in [15]. Compared with the existing literature, our methods allow to reduce the regularity required of the initial data to the case whose existence couldn't be implied by classical facts. The result relies on the use of additional L^∞-estimates.

Let $x=(x_1,\dots,x_n)$, $\partial_x=(\partial_{x_1},\dots,\partial_{x_n})$,

$$P(t,x;\partial_t,\partial_x)=a_0\partial_t^2+\sum_{i=1}^n a_i\partial_{tx_i}^2+\sum_{i,j=1}^n b_{ij}\partial_{x_ix_j}^2 \tag{1.1}$$

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is a second order strictly hyperbolic operator with respect to t, its coefficients are smooth functions in $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \{t \geqslant 0\}$ with $b_{ij} = b_{ji}$.

At first, consider the following special Cauchy problem in $\mathbb{R}^3_+$:

$$\begin{cases} P(t,x;\partial_t,\partial_x)u = f(t,x,u) \\ u|_{t=0} = 0 \\ u_t|_{t=0} = u_1(x) \end{cases} \tag{1.2}$$

where f is smooth with respect to its arguments and other notations are given as above with $n=2$. For this problem, we have the following result:

Theorem 1.1. *Let $\mathscr{C}$ be the forward characteristic cone issuing from the origin for P, $k \geqslant 0$ be an integer, ω be a neighborhood of the origin on $\{t=0\}$. If $u_1(x) \in L^\infty \cap N^k_{\{0\}}(\omega)$, then there exists a unique solution $u \in L^\infty \cap N^{1,k}_{\mathscr{C}}(\Omega)$ to* (1.2), *where Ω is a small determinacy domain of ω with respect to P in R^3_+, $N^k_{\{0\}}(\omega)$ and $N^{1,k}_{\mathscr{C}}(\Omega)$ are conormal distribution spaces defined as in* [15] (*Their definitions will be recalled in Sect.* 2.).

Remark 1.1. From the proof of this theorem, which will be given in Sect. 4, we can easily obtain that the above result is also valid for the case of the operator P containing some linear lower order terms.

Let us discuss the general Cauchy problem for the fully semilinear second order hyperbolic equation in $R^{n+1}_+ = R^{n+1} \cap \{t \geqslant 0\}$ with $n \geqslant 2$ as follows:

$$\begin{cases} P(t,x;\partial_t,\partial_x)u = f(t,x,u,\nabla u) \\ u|_{t=0} = u_0(x) \\ u_t|_{t=0} = u_1(x) \end{cases} \tag{1.3}$$

with P being given in (1.1). For this problem, using the idea of [13] for the choice of the solution space, we have

Theorem 1.2. *Let $\mathscr{C}$ be given as in Theorem* 1.1, *$k \geqslant 0$ be an integer. Given any $s > n/2$, if $u_i(x) \in H^{s+1-i} \cap N^{1-i,k}_{\{0\}}(\omega)$ $(i=0,1)$, then* (1.3) *admits a unique solution $u \in W^{1,\infty} \cap N^{1,k}_{\mathscr{C}}(\Omega)$, where $H^{s+1-i}(\omega)$ and $W^{1,\infty}(\Omega)$ are Sobolev spaces based on $L^2(\omega)$ and $L^\infty(\Omega)$ respectively, and all other notations are defined as in Theorem* 1.1.

Remark 1.2. If the nonlinear term $F(t,x,u,\nabla u)$ of (1.3) does not depend on ∇u, and the initial condition of Theorem 1.2 is weakened as

$$u_i(x)\in H^{s-i}\cap N^{1-i,k}_{\{0\}}(\omega) \quad (i=0,1) \tag{1.4}$$

with any $s>\frac{n}{2}$, then the problem (1.3) also admits a unique solution u in $L^\infty\cap N^{1,k}_{\mathscr{C}}(\Omega)$, where every notation is the same as in Theorem 1.2.

The remainder of this paper is arranged as follows. In Sect. 2, we will introduce the definition and some properties of conormal distribution spaces, and establish a result for the symbol of the operator P as a preparation by using an argument similar to [2]. In Sect. 3, we discuss the linear problem corresponding to (1.2) or (1.3), and establish several estimates. Then, by using estimates obtained in Sect. 3, we deal with the nonlinear problem (1.2), and obtain the proof of Theorem 1.1 in Sect. 4. Finally, by combining some well-known classical results with estimates of Sect. 3 it follows the conclusions of Theorem 1.2 and Remark 1.2 in Sect. 5.

2. Some Basic Results

Without loss of generality, we can suppose that the forward characteristic cone $\mathscr{C}$ issuing from the origin with respect to P has the standard form $\mathscr{C}=\{t=|x|>0\}$ by using a proper transformation (The proof of this fact can be found in [7].). As in [15], we know that the algebra $\mathscr{M}$ of vectors tangential to $\mathscr{C}$ and $\{0\}$ in R^{n+1}_+ can be generated by

$$M_0,\quad \{M_h\}_{h=1}^n,\quad \{M_{js}\}_{1\leqslant j<s\leqslant n} \tag{2.1}$$

where $M_0=t\partial_t+\sum_{i=1}^n x_i\partial_{x_i}$, $M_h=t\partial_{x_h}+x_h\partial_t$ and $M_{js}=x_j\partial_{x_s}-x_s\partial_{x_j}$, the algebra $\mathscr{M}_0$ of vectors tangential to $\{0\}$ in R^n is generated by

$$\{x_i\partial_{x_j}\}_{1\leqslant i,j\leqslant n}. \tag{2.2}$$

Definition 2.1. Let Ω (ω resp.) be a domain of R^{n+1}_+ (R^n resp.), $\mathscr{S}$ be a space of vector fields in Ω (ω resp.), $\{M_1,\dots,M_k\}$ be generators of $\mathscr{S}$, $p\in R$, $q\geqslant 0$ be an integer. We define

$$N^{p,q}(\Omega;\mathscr{S})=\{u\in H^p(\Omega)\,|\,M^I u\in H^p(\Omega),\quad \text{where } M^I=M_1^{i_1}\cdots M_k^{i_k},$$
$$|I|=i_1+\cdots+i_k\leqslant q\}$$

($N^{p,q}(\omega;\mathscr{S})$ resp.), and its norm

$$\|u\|_{N^{p,q}(\Omega,\mathscr{S})}=\left\{\sum_{|I|\leqslant q}\|M^I u\|^2_{H^p(\Omega)}\right\}^{1/2}.$$

Set $N^{p,q}_{\mathscr{C}}(\Omega)=N^{p,q}(\Omega;\mathscr{M})$, $N^q_{\mathscr{C}}(\Omega)=N^{0,q}_{\mathscr{C}}(\Omega)$, $N^{p,q}_{\{0\}}(\omega)=N^{p,q}(\omega;\mathscr{M}_0)$ and $N^q_{\{0\}}(\omega)=N^{0,q}_{\{0\}}(\omega)$.

For the above conormal distribution spaces, we have

Lemma 2.1 (*see* [11], *Lemma* 5.2). *If $p,q\geqslant 0$ are integers, then there is a contant $C>0$ such that for any $u\in L^\infty\cap N^{p,q}_{\mathscr{C}}(\Omega)$, the estimate*

$$\|\partial^\alpha M^\beta u\|_{L^s(\Omega)}\leqslant C(\|u\|_{L^\infty(\Omega)}+\|u\|_{N^{p,q}_{\mathscr{C}}(\Omega)}) \tag{2.3}$$

holds, where $\partial^\alpha=\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$ with $|\alpha|=\sum_{i=0}^n\alpha_i\leqslant p$, $M^\beta=M_1^{\beta_1}\cdots M_k^{\beta_k}$ with $|\beta|=\sum_{i=1}^k\beta_i\leqslant q$ and $M_i\in\mathscr{M}$ $(i=1,\dots,k)$ and $s=\dfrac{2(p+q)}{|\alpha|+|\beta|}$.

Proposition 2.1. *Suppose a, $u\in L^\infty\cap N^{p,q}_{\mathscr{C}}(\Omega)$ for integers p, $q\geqslant 0$.*

(1) *The estimate*

$$\|au\|_{N^{p,q}_{\mathscr{C}}(\Omega)}\leqslant C(\|u\|_{L^\infty(\Omega)}+\|u\|_{N^{p,q}_{\mathscr{C}}(\Omega)}) \tag{2.4}$$

holds, where the constant C depends only on the norm of a in $L^\infty\cap N^{p,q}_{\mathscr{C}}(\Omega)$.

(2) *If $f\in C^\infty(\mathbb{R})$, then $f(u)\in N^{p,q}_{\mathscr{C}}(\Omega)$; moreover, we have*

$$\|f(u)\|_{N^{p,q}_{\mathscr{C}}(\Omega)}\leqslant F(\|u\|_{L^\infty(\Omega)})\sup_{1\leqslant k\leqslant p+q}(\|u\|_{L^\infty(\Omega)}+\|u\|_{N^{p,q}_{\mathscr{C}}(\Omega)})^k \tag{2.5}$$

with the function $F(r)=\sup_{0\leqslant\bar r\leqslant r}\sum_{i=0}^{p+q}|f^{(i)}(\bar r)|$.

Proof. (1) For any ∂^α, M^β indicated as in Lemma 2.1, we have

$$\|\partial^\alpha M^\beta(au)\|_{L^2}\leqslant\sum_{|\alpha_1|+|\alpha_2|=|\alpha|,|\beta_1|+|\beta_2|=|\beta|}\|(\partial^{\alpha_1}M^{\beta_1}a)(\partial^{\alpha_2}M^{\beta_2}u)\|_{L^2} \tag{2.6}$$

From Lemma 2.1, we know that $\partial^{\alpha_1}M^{\beta_1}a\in L^{\frac{2(|\alpha|+|\beta|)}{|\alpha_1|+|\beta_1|}}$ and $\partial^{\alpha_2}M^{\beta_2}u\in L^{\frac{2(|\alpha|+|\beta|)}{|\alpha_2|+|\beta_2|}}$. Hölder's inequality implies that (2.6) can be controlled by the right hand side of (2.4).

(2) The Leibniz formula implies that $\partial^\alpha M^\beta(f(u))$ may be written as a sum of terms of the form

$$g_{\alpha\beta}(u)\partial^{\alpha^1}M^{\beta^1}u\cdots\partial^{\alpha^m}M^{\beta^m}u$$

with $g_{\alpha\beta}$ smooth, $\sum_{i=1}^m|\alpha^i|=|\alpha|$ and $\sum_{i=1}^m|\beta^i|=|\beta|$. Again by applying

Hölder's inequality and Lemma 2.1 for this term, we obtain the estimate (2.5). ■

Lemma 2.2. *If* $w(x)\in N^{s,k}_{\{0\}}(\omega)$, *then* $x_i w$ *belongs to* $N^{s+1,k-1}_{\{0\}}(\omega)$ *for any* $s\geqslant 0$, *integer* $k\geqslant 1$ *and* $i\in\{1,\dots,n\}$; *moreover, there is a constant* $C>0$ *such that for any* $w(x)\in N^{s,k}_{\{0\}}(\omega)$, *the following estimate*

$$\| x_i w\|_{N^{s+1,k-1}_{\{0\}}(\omega)} \leqslant C\| w\|_{N^{s,k}_{\{0\}}(\omega)} \tag{2.7}$$

is valid.

Since

$$\partial_{x_i}(x_j w) = \delta_{ij} w + x_j \partial_{x_i} w$$

with δ_{ij} being the Kronecker symbol, this lemma is immediately deduced from (2.2) and Definition 2.1.

Next, we give a commutator result which will be used in Sect. 3.

Lemma 2.3 (*see* [15]). *Let* $\Box = \partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2$ *be the d'Alembertien operator, then*

$$\begin{cases} [M_0,\Box] = -2\Box, \quad [M_h,\Box]=0, \quad [M_{js},\Box]=0 \\ [M_0,M_h]=0, \quad [M_0,M_{js}]=0, \\ [M_h,M_{js}] = \delta_{hj}M_s - \delta_{hs}M_j \quad [M_h,M_k] = M_{hk}, \\ [M_{js},M_{pq}] = \delta_{ps}M_{jq} + \delta_{qj}M_{sp} + \delta_{qs}M_{pj} + \delta_{jp}M_{qs} \end{cases} \tag{28}$$

where $h,k\in\{1,\dots,n\}$, $1\leqslant j<s\leqslant n$, $1\leqslant p<q\leqslant n$, *all vectors are given in* (2.1) *and we set* $M_{j_1s_1} = -M_{s_1j_1}$ *as* $1\leqslant s_1<j_1\leqslant n$.

Since the operator P given in (1.1) is strictly hyperbolic with respect to t, we can always assume $a_0\equiv 1$ of P in the following discussion. Let $p(t,x;\tau,\xi) = \tau^2 + \sum_{i=1}^n a_i\tau\xi_i + \sum_{i,j=1}^n b_{ij}\xi_i\xi_j$ denote the symbol of $-P$, $m_0 = t\tau + \sum_{i=1}^n x_i\xi_i$, $m_h = t\xi_h + x_h\tau$ $(h=1,\dots,n)$ and $m_{js} = x_j\xi_s - x_s\xi_j$ $(1\leqslant j<s\leqslant n)$ with $\xi=(\xi_1,\dots,\xi_n)$ which represent the respective symbols of tangential vectors introduced in (2.1) with respect to $\mathscr{C}$. For this symbol $p(t,x;\tau,\xi)$, by using an argument similar to [2] we get a decomposition result as follows:

Proposition 2.2. *For this* $p(t,x;\tau,\xi)$, *the following decomposition*

$$p = \tau^2 - \sum_{i=1}^n \xi_i^2 + \sum_{h=0}^n p_h m_h + \sum_{1\leqslant j<s\leqslant n} p_{js}m_{js} \tag{2.9}$$

is valid, where p_h *and* p_{js} *are all symbols of order one.*

Proof. Here, we only prove (2.9) for the case $n=2$, the general case can be similarly proved. First, we denote the relation $A-B=C\alpha$ for any two smooth functions A and B by $A\sim B$, where $\alpha=t^2-x_1^2-x_2^2$ and C is an arbitrary smooth function.

Since the normal vector of $\mathscr{C}$ is $(t,-x_1,-x_2)$ at $(t,x_1,x_2)\in\mathscr{C}$, and $\mathscr{C}$ is the lightcone of P issuing from the origin, we have

$$p(t,x;t,-x)=t^2+b_{11}x_1^2+b_{22}x_2^2-a_1tx_1+2b_{12}x_1x_2-a_2x_2t\sim 0. \tag{2.10}$$

Obviously, we have $t^2\sim x_1^2+x_2^2$, which substituted into (2.10) gives that there is a smooth function C_1 such that

$$[(1+b_{11})x_1-a_1t+2b_{12}x_2]x_1+[(1+b_{22})x_2-a_2t]x_2=C_1\alpha. \tag{2.11}$$

Therefore, there are two smooth functions C_2 and C_3 such that $C_1=C_2x_1+C_3x_2$. Now, (2.11) is reduced to

$$\begin{aligned}[(1+b_{11})x_1-a_1t+2b_{12}x_2-C_2\alpha]x_1+\\+[(1+b_{22})x_2-a_2t-C_3\alpha]x_2=0\end{aligned} \tag{2.12}$$

which implies that there is a smooth function C_4 such that

$$(1+b_{22})x_2-a_2t\sim C_4x_1. \tag{2.13}$$

From (2.13), we know that there are two smooth functions C_5 and C_6 such that

$$1+b_{22}\sim C_5x_1+C_6t. \tag{2.14}$$

On the other hand, (2.13) implies that as $x_2\neq 0$, we have

$$1+b_{22}\sim\frac{a_2}{x_2}t+\frac{C_1}{x_2}x_1.$$

Comparing this with (2.14) we deduce that there is a function C_7 which is smooth in $\{x_2\neq 0\}$, such that

$$a_2\sim C_6x_2+C_7x_2x_1. \tag{2.15}$$

Since a_2 and C_6x_2 are all smooth at $\{x_2=0\}$, we have $\bar{C}_7=C_7x_2$ is smooth at $\{x_2=0\}$ which implies $\bar{C}_7$ smooth in R^3_+ since $\bar{C}_7$ is smooth in $\{x_2\neq 0\}$. Substituting (2.14) and (2.15) into (2.11), we obtain

$$(1+b_{11})x_1-a_1t+C_8x_2\sim 0 \tag{2.16}$$

with $C_8=2b_{12}+C_5x_2-\bar{C}_7t$.

Therefore, there exist two smooth functions C_9 and C_{10} such that

$$1 + b_{11} \sim C_9 x_2 + C_{10} t. \tag{2.17}$$

But, (2.16) implies that as $x_1 \neq 0$

$$1 + b_{11} \sim \frac{a_1}{x_1} t - \frac{C_8}{x_1} x_2. \tag{2.18}$$

Comparing this with (2.17) it follows that there is a function C_{11} which is smooth in $\{x_1 \neq 0\}$, such that

$$a_1 \sim C_{10} x_1 + C_{11} x_1 x_2. \tag{2.19}$$

Since a_1 and $C_{10} x_1$ are smooth at $\{x_1 = 0\}$, we have $\bar{C}_{11} = C_{11} x_1$ being also smooth at $\{x_1 = 0\}$ which implies that $\bar{C}_{11}$ is smooth in R^3_+ since $\bar{C}_{11}$ is smooth in $\{x_1 \neq 0\}$. Substituting (2.19) into (2.18), and using (2.17) we get

$$C_8 \sim \bar{C}_{11} t - C_9 x_1. \tag{2.20}$$

But $C_8 = 2b_{12} + C_5 x_2 - \bar{C}_7 t$, so

$$b_{12} \sim (\bar{C}_{11} t + \bar{C}_7 t - C_9 x_1 - C_5 x_2)/2. \tag{2.21}$$

Substituting (2.14), (2.15), (2.17), (2.19) and (2.21) into p, and noting that the term $\alpha \bar{p}_2$ can always be expressed as a sum of the third and fourth terms of the right hand side of (2.9) for any second order symbol $\bar{p}_2$, we obtain our conclusion by simple computation. ■

3. The Linear Problem

In the following discussion we shall always use C or C_Δ to denote some constants independent of k, T and u with Δ being an integer, assume that Ω is the domain of the determinacy of ω with respect to P with ω being given in Theorem 1.1, and set $\Omega_T = \Omega \cap \{t < T\}$ for any $T > 0$ unless otherwise stated.

We first give a lemma, which will be used later, to point out the regularity of solutions to the Cauchy problem of strictly hyperbolic linear equations (we refer to [6], page 370–371).

Lemma 3.1. *Suppose that* $P(t, x, \partial_t, \partial_x) = \partial_t^m - \sum_{j=0}^{m-1} A_{m-j} \partial_t^j$ *is strictly hyperbolic with respect to t in* $R_t \times R^n$, *where* $A_{m-j} = a_{m-j}(t, x, \partial_x)$ *is a pseudodifferential operator with order* $m - j$ *and independent of* (t, x) *for* $|x|$ *sufficiently large. Let* $s \in \mathbb{R}$, $f \in L^2([0, T]; H^s)$, $g_j \in H^{s+m-j}$

with $j=1,\dots,m$, *then there exists a unique* $u\in\bigcap_{k=0}^{m-1}C^k([0,T];H^{s+m-1-k})$ *such that* $Pu=f$ *in* $\mathscr{D}'((0,T)\times\mathbb{R}^n)$, $\partial_t^{j-1}u(0,\cdot)=g_j$. *Furthermore, there exists C such that*

$$\sum_{j=1}^{m}\|\partial_t^{j-1}u(T,\cdot)\|_{s+m-j}^2\leqslant C\left\{\sum_{j=1}^{m}\|g_j\|_{s+m-j}^2+\int_0^T\|f(t,\cdot)\|_s^2\,dt\right\}. \tag{3.1}$$

In this section, we consider the following problem in $\mathbb{R}_+^{n+1}$:

$$\begin{cases}P(t,x;\partial_t,\partial_x)v=g(t,x)\\ v|_{t=0}=u_0(x)\\ v_t|_{t=0}=u_1(x)\end{cases} \tag{3.2}$$

where P is given as in (1.1). For this problem we have

Proposition 3.1. *Given any* $s\geqslant 0$, *and integer* $k\geqslant 0$. *If* $u_i\in N_{\{0\}}^{s+1-i,k}(\omega)$ $(i=0,1)$ *and* $g\in N_{\mathscr{C}}^{s,k}(\Omega_T)$, *then there is a unique solution* $v\in N_{\mathscr{C}}^{s+1,k}(\Omega_T)$ *to* (3.2); *moreover*, the estimate

$$\|v\|_{N_{\mathscr{C}}^{s+1,k}(\Omega_T)}\leqslant C_1\sqrt{T}\left(\|u_0\|_{N_{\{0\}}^{s+1,k}(\omega)}+\|u_1\|_{N_{\{0\}}^{s,k}(\omega)}+\|g\|_{N_{\mathscr{C}}^{s,k}(\Omega_T)}\right) \tag{3.3}$$

holds.

Before the proof of this proposition, we first give

Lemma 3.2. *Given any* $s\geqslant 0$, *if* $u_i\in H^{s+1-i}(\omega)$ $(i=0,1)$ *and* $g\in H^s(\Omega_T)$, *then the solution* v *of* (3.2) *constructed in Lemma* 3.1 *belongs to* $H^{s+1}(\Omega_T)$; *moreover, the estimate*

$$\|v\|_{H^{s+1}(\Omega_T)}\leqslant C_2\sqrt{T}\left(\|u_0\|_{H^{s+1}(\omega)}+\|u_1\|_{H^s(\omega)}+\|g\|_{H^s(\Omega_T)}\right) \tag{3.4}$$

is valid.

Proof. Applying the classical result Lemma 3.1 for (3.2) immediately implies the estimate

$$\sum_{j=0}^{1}\|D_t^jv(t_1,\cdot)\|_{H^{s+1-j}(\bar\Omega_{t_1})}^2\leqslant$$

$$\leqslant C\left\{\sum_{j=0}^{1}\|u_j\|_{H^{s+1-j}(\omega)}^2+\int_0^{t_1}\|g(t,\cdot)\|_{H^s(\bar\Omega_t)}^2\,dt\right\} \tag{3.5}$$

holds for the solution v of (3.2), where $\bar\Omega_{t_1}=\Omega\cap\{t=t_1\}$ with $t_1>0$. Integrating the above relation with respect to t_1 over $[0,T]$ it follows that $\|\partial_tv\|_{L^2([0,T],H^s(\bar\Omega_t))}$ and $\|v\|_{L^2([0,T],H^{s+1}(\bar\Omega_t))}$ can be controlled by the right hand side of (3.4), which implies that the estimate (3.4) is valid for the case of $s = 0$.

If $s \geqslant 1$ is an integer, by using (3.5) and the equation of (3.2) we obtain that for any $2 \leqslant j \leqslant s + 1$, $\| \partial_t^j v \|^2_{L^2([0,T],H^{s+1-j}(\bar{\Omega}_t))}$ is controlled by the right hand side of (3.4). Therefore, the relation (3.4) is valid for the case of s being integer. That is to say that the linear operator $(u_0, u_1, g) \mapsto v$ defined by the problem (3.2) is bounded from $H^{s+1}(\omega) \times \times H^s(\omega) \times H^s(\Omega_T)$ to $H^{s+1}(\Omega_T)$, and its norm is less or equal to $C_2\sqrt{T}$ for any $T > 0$ and integer $s \geqslant 0$. Applying the usual interpolation theory (see [14]) for this operator we deduce that this operator is also bounded between the above two spaces for any $s \geqslant 0$, and its norm still can be controlled by $C_2\sqrt{T}$, which is equivalent to the estimate (3.4) holding for the solution v to the problem (3.2). ■

Proof of Proposition 3.1. For simplicity, we only prove the case $n = 2$, and it is obvious that the general case can be similarly proved.

From Lemma 3.2, we know that the estimate (3.3) holds for the case $k = 0$. Now, we want to prove the case $k = 1$. Let $P_0(t, x; \partial_t, \partial_x)$ ($P_1(t, x; \partial_t, \partial_x)$, $P_2(t, x; \partial_t, \partial_x)$ and $P_{12}(t, x; \partial_t, \partial_x)$ resp.) be the differential operator corresponding to the symbol ip_0 (ip_1, ip_2 and ip_{12} resp.) which is given in (2.9), then Proposition 2.2 implies

$$P = \square + \sum_{i=0}^{2} P_i M_i + P_{12} M_{12} \tag{3.6}$$

with $\square = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$ being the d'Alembertien operator and all vectors being given in (2.1).

Using Lemma 2.3 and (3.6) there is a 4×4 matrix Q of first order differential operators such that

$$\mathscr{P} V + QV = G \tag{3.7}$$

with $\mathscr{P} = \operatorname{diag}(P,\ P,\ P,\ P)$, $V = (M_0 v,\ M_1 v,\ M_2 v,\ M_{12} v)^T$ and $G = ((M_0 + 2)g,\ M_1 g,\ M_2 g,\ M_{12} g)^T$. Under the assumption of this proposition, we have: $V|_{t=0} \in H^{s+1}(\omega)$, $V_t|_{t=0} \in H^s(\omega)$ by using Lemma 2.2. For the equation (3.7), again using Lemma 3.2 we get the estimate (3.3) for the case $k = 1$. In the same way, we can prove (3.3) for any integer k, which implies that the solution v of (3.2) belongs to $N^{s+1,k}_{\mathscr{C}}(\Omega_T)$. ■

Let us consider the L^∞-boundedness of the solution v to (3.2) for the particular case $u_0 \equiv 0$ and $n = 2$.

Proposition 3.2. *Consider the following linear problem in* $\mathbb{R}^3_+$:

$$\begin{cases} P(t,x;\partial_t,\partial_x)v = g(t,x) \\ v|_{t=0} = 0 \\ v_t|_{t=0} = u_1(x) \end{cases} \tag{3.8}$$

with $x=(x_1,x_2)$ *and* P *given as in* (1.1). *If* $g\in L^\infty(\Omega_T)$ *and* $u_1\in L^\infty(\omega)$, *then the solution* v *satisfies the estimate*

$$\|v\|_{L^\infty(\Omega_T)} \leqslant C_3(\|u_1\|_{L^\infty(\omega)} + T\|g\|_{L^\infty(\Omega_T)}). \tag{3.9}$$

Proof. From the Book III of [10], we know that the solution v of the problem (3.8) can be written in the form:

$$v(t,x_1,x_2) = \iint_{\sigma_0} G(t,x_1,x_2;0,\xi_1,\xi_2)u_1(\xi_1,\xi_2)d\xi_1 d\xi_2 +$$

$$+ \int_0^t \iint_{\sigma_\tau} G(t,x_1,x_2;\tau,\xi_1,\xi_2)g(\tau,\xi_1,\xi_2)d\xi_1 d\xi_2 d\tau \tag{3.10}$$

where $\sigma_\tau (0\leqslant\tau\leqslant t)$ is the domain of the intersection of $\{t=\tau\}$ with the backward characteristic cone with vertex at (t,x_1,x_2), G is the elementary solution with the form

$$G(t,x_1,x_2;\tau,\xi_1,\xi_2) = \frac{H}{\Gamma^{1/2}}. \tag{3.11}$$

In (3.11), H is a smooth function of $(t,x_1,x_2;\tau,\xi_1,\xi_2)$, $\Gamma=0$ is a backward conic surface with vertex at (t,x_1,x_2), $\nabla\Gamma\neq 0$ except $(\tau,\xi_1,\xi_2)=(t,x_1,x_2)$. Particularly, when P is the wave operator $\partial_t^2-\partial_{x_1}^2-\partial_{x_2}^2$, we have $H=1$ and $\Gamma=(t-\tau)^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2$.

Obviously, from (3.11) we know that G is an integrable function with respect to $(\xi_1,\xi_2)\in\sigma_\tau$ for any $\tau\in[0,t]$. Therefore, (3.10) immediately implies

$$\|v\|_{L^\infty(\Omega_T)} \leqslant C(\|u_1\|_{L^\infty(\omega)} + T\|g\|_{L^\infty(\Omega_T)}). \quad \blacksquare$$

4. Proof of Theorem 1.1

Without loss of generality, we will assume that $f(t,x,0)=0$ in this section. For the problem (1.2), we construct an iterative sequence $\{u^\nu\}_{\nu=0}^\infty$ by the following process:

$$\begin{cases} P(t,x,\partial_t,\partial_x)u^{v+1} = f(u^v), & \text{in } \mathbb{R}^3_+ \\ u^{v+1}|_{t=0} = 0 \\ u_t^{v+1}|_{t=0} = u_1 \end{cases} \tag{4.1}$$

with u^0 satisfying

$$\begin{cases} P(t,x,\partial_t,\partial_x)u^0 = 0, & \text{in } \mathbb{R}^3_+ \\ u^0|_{t=0} = 0 \\ u_t^0|_{t=0} = u_1 \end{cases} \tag{4.2}$$

where we already write $f(t,x,u)$ as $f(u)$ for simplicity.

By using Propositions 2.1, 3.1 and 3.2, we know that for any $v \in \{0,1,2,\dots\}$, the solution u^v of (4.1), (4.2) belongs to $L^\infty \cap N^{1,k}_{\mathscr{C}}(\Omega_T)$ for any $T > 0$ under the assumption of Theorem 1.1. For this sequence, we have

Proposition 4.1. *Given any $T_0 > 0$, there is $T_1 \in (0, T_0]$ such that the sequence $\{u^v\}_{v=0}^\infty$ is bounded in $L^\infty \cap N^{1,k}_{\mathscr{C}}(\Omega_{T_1})$.*

Proof. Let $R_1 = \|u_1\|_{N^k_{\{0\}}(\omega)}$, $r_1 = C_3\|u_1\|_{L^\infty(\omega)}$ with C_3 given as in (3.9), $r = \max(r_1 + 1, \|u^0\|_{L^\infty(\Omega_{T_0})})$ and $R = \|u^0\|_{N^{1,k}_{\mathscr{C}}(\Omega_{T_0})} + 1$, then we want to prove that there is a $T_1 \in (0, T_0]$ such that

$$\|u^v\|_{L^\infty(\Omega_{T_1})} \leqslant r \quad \text{and} \quad \|u^v\|_{N^{1,k}_{\mathscr{C}}(\Omega_{T_1})} \leqslant R \tag{4.3}$$

for any $v = 0,1,2,\dots$, which we will prove by induction over v.

From the above choice of R and r, we know that (4.3) is obviously satisfied for the case $v = 0$. Assuming (4.3) valid for v, it is enough to prove it for the case $v + 1$.

Since (4.3) holds for u^v, using Proposition 2.1 yields

$$\begin{aligned} \|f(u^v)\|_{N^k_{\mathscr{C}}(\Omega_T)} \leqslant F(\|u^v\|_{L^\infty(\Omega_T)}) \sup_{1 \leqslant k_1 \leqslant k} (\|u^v\|_{L^\infty(\Omega_T)} + \\ + \|u^v\|_{N^k_{\mathscr{C}}(\Omega_T)})^{k_1} \leqslant F(r)(r+R)^k \end{aligned} \tag{4.4}$$

where $F(r) = \sup_{\bar r \in [0,r]} \sum_{|\alpha| \leqslant k} |f^{(\alpha)}(\bar r)|$.

Let $T_1 \in (0, T_0]$ satisfy

$$\begin{cases} C_1\sqrt{T_1}(R_1 + F(r)(r+R)^k) \leqslant R \\ r_1 + C_3 T_1 \sup_{\bar r \in [0,r]} |f'(\bar r)| r \leqslant r \end{cases} \tag{4.5}$$

with C_1 and C_3 being constants indicated as in Propositions 3.1 and 3.2 respectively.

For the problem (4.1), using (3.3) and (3.9) it follows that (4.3) holds for the case $\nu+1$ from the choice of T_1 in (4.5). ■

Proposition 4.2. *For T_1 given in Proposition 4.1, there is $T_2 \in (0, T_1]$ such that the sequence $\{u^\nu\}_{\nu=0}^\infty$ is convergent in $L^\infty \cap N_{\mathscr{C}}^{1,k}(\Omega_{T_2})$.*

Proof. For any $\nu \in \{1, 2, 3, \ldots\}$, $u^{\nu+1} - u^\nu$ satisfies

$$\begin{cases} P(u^{\nu+1} - u^\nu) = f(u^\nu) - f(u^{\nu-1}) \\ (u^{\nu+1} - u^\nu)|_{t=0} = 0 \\ (u^{\nu+1} - u^\nu)_t|_{t=0} = 0. \end{cases} \tag{4.6}$$

The L^∞-boundedness of $\{u^\nu\}_{\nu=0}^\infty$ in Ω_{T_1} implies that for any $T \in (0, T_1]$

$$\| f(u^\nu) - f(u^{\nu-1}) \|_{L^\infty(\Omega_T)} \leqslant C \| u^\nu - u^{\nu-1} \|_{L^\infty(\Omega_T)} \tag{4.7}$$

is valid, and the boundedness of $\{u^\nu\}_{\nu=0}^\infty$ in $L^\infty \cap N_{\mathscr{C}}^{1,k}(\Omega_{T_1})$ implies that for any $T \in (0, T_1]$,

$$\begin{aligned} \| f(u^\nu) - f(u^{\nu-1}) \|_{N_{\mathscr{C}}^k(\Omega_T)} \leqslant C(\| u^\nu - u^{\nu-1} \|_{L^\infty(\Omega_T)} + \\ + \| u^\nu - u^{\nu-1} \|_{N_{\mathscr{C}}^k(\Omega_T)}) \end{aligned} \tag{4.8}$$

holds by using the mean value theorem and Proposition 2.1(a).

Therefore, by applying Proposition 3.1 and 3.2 to the problem (4.6) we obtain that for any $T \in (0, T_1]$,

$$\| u^{\nu+1} - u^\nu \|_{L^\infty(\Omega_T)} \leqslant CT \| u^\nu - u^{\nu-1} \|_{L^\infty(\Omega_T)} \tag{4.9}$$

and

$$\begin{aligned} \| u^{\nu+1} - u^\nu \|_{N_{\mathscr{C}}^{1,k}(\Omega_T)} \leqslant C\sqrt{T}(\| u^\nu - u^{\nu-1} \|_{N_{\mathscr{C}}^k(\Omega_T)} + \\ + \| u^\nu - u^{\nu-1} \|_{L^\infty(\Omega_T)}) \end{aligned} \tag{4.10}$$

are valid by using (4.7) and (4.8).

Combining (4.9) with (4.10) yields our conclusion of this proposition. ■

From Propositions 4.1 and 4.2 we immediately obtain the conclusion of Theorem 1.1.

5. Proof of Theorem 1.2

Without loss of generality, we will always consider the problem (1.3) in $t \in [0, T_0]$ with T_0 being chosen so that $\bar{\Omega}_{T_0} = \Omega \cap \{t = T_0\}$ is a non-void connected open domain, and we suppoose that there is a sufficiently regular diffeomorphism $\chi:\Omega_{T_0}\to[0,T_0]\times\bar{\Omega}_{T_0}$ such that χ preserves $\{t=\text{constant}\}$. For convenience, we introduce

Definition 5.1. The space $C^m([0,T];H^s(\bar{\Omega}_t))$ is defined by transformation to $[0,T]\times\bar{\Omega}_{T_0}$, i.e. $u\in C^m([0,T];H^s(\bar{\Omega}_t))$ iff $u\circ\chi^{-1}\in C^m([0,T];H^s(\bar{\Omega}_{T_0}))$, where $m\geqslant 0$, $\bar{\Omega}_\tau$ denotes $\Omega\cap\{t=\tau\}$ with any $\tau\in(0,T]$ and $T\leqslant T_0$.

Before the discussion of the problem (1.3), we first give a well-known Sobolev embedding theorem as follows; its proof can be found in any books on Sobolev spaces.

Lemma 5.1. *Given any $s>\frac{n}{2}$, for any $u\in C([0,T],H^s(\bar{\Omega}_t))$, we have $u\in L^\infty(\Omega_T)$ and*

$$\|u\|_{L^\infty(\Omega_T)}\leqslant C\|u\|_{C([0,T],H^s(\bar{\Omega}_t))} \tag{5.1}$$

with C independent of $T\in(0,T_0]$.

As in Sect. 4, for the problem (1.3) we construct an iterative sequence $\{u^\nu\}_{\nu=0}^{\infty}$ by the following process:

$$\begin{cases} P(t,x,\partial_t,\partial_x)u^{\nu+1}=f(u^\nu,\nabla u^\nu), & \text{in } \mathbb{R}_+^{n+1}\\ u^{\nu+1}|_{t=0}=u_0 \\ u_t^{\nu+1}|_{t=0}=u_1 \end{cases} \tag{5.2}$$

with u^0 satisfying

$$\begin{cases} P(t,x,\partial_t,\partial_x)u^0=0, & \text{in } \mathbb{R}_+^{n+1}\\ u^0|_{t=0}=u_0 \\ u_t^0|_{t=0}=u_1 \end{cases} \tag{5.3}$$

where we already write $f(t,x,u,\nabla u)$ as $f(u,\nabla u)$, and suppose $f(t,x,0,0)=0$ for simplicity.

Given any $s>\frac{n}{2}$, from now on we will always suppose that the assumption of Theorem 1.2 is valid for the problem (5.2), (5.3).

From Lemma 3.1 we know that the initial condition $u_i(x)\in\in H^{s+1-i}(\omega)$ $(i=0,1)$ yields that the solution u^0 of (5.3) belongs to $C([0,T],H^{s+1}(\bar{\Omega}_t))\cap C^1([0,T],H^s(\bar{\Omega}_t))$ for any $T\in(0,T_0]$, which implies $u^0\in W^{1,\infty}(\Omega_T)$ by using Lemma 5.1.

As in Lemma 1.5 of [3], we have that for any smooth function f, there is a function G which depends only on f such that

$$\| f(u^0, \nabla u^0) \|_{H^s(\bar{\Omega}_t)} \leqslant G(|||u^0|||_{s,T}) \tag{5.4}$$

uniformly in $t \in [0, T]$, where $|||\cdot|||_{s,T}$ denotes the norm of $C([0,T], H^{s+1}(\bar{\Omega}_t)) \cap C^1([0,T], H^s(\bar{\Omega}_t))$. Therefore, we deduce that $f(u^0, \nabla u^0)$ belongs to $L^2([0,T], H^s(\bar{\Omega}_t))$.

Again using Lemma 3.1 implies $u^1(x) \in C([0,T], H^{s+1}(\bar{\Omega}_t)) \cap \cap C^1([0,T], H^s(\bar{\Omega}_t))$ for the problem (5.2) with $\nu = 0$. By the above procedure we conclude that for any $\nu \in \{0, 1, 2, \ldots\}$, the solution u^ν of (5.2), (5.3) belongs to $C([0,T], H^{s+1}(\bar{\Omega}_t)) \cap C^1([0,T], H^s(\bar{\Omega}_t))$, which implies $u^\nu \in W^{1,\infty}(\Omega_T)$ as above.

The condition $u_i(x) \in N^{1-i,k}_{\{0\}}(\omega)$ $(i = 0, 1)$ implies that the solution $u^0 \in N^{1,k}_{\mathscr{C}}(\Omega_T)$ by using Proposition 3.1. Hence $f(u^0, \nabla u^0)$ belongs to $N^k_{\mathscr{C}}(\Omega_T)$ by using Proposition 2.1 since we already have $u^0 \in W^{1,\infty}(\Omega_T)$ above. Again applying Proposition 3.1 to the problem (5.2) with $\nu = 0$ it follows $u^1 \in N^{1,k}_{\mathscr{C}}(\Omega_T)$. By this procedure we obtain that for any $\nu \in \{0, 1, 2, \ldots\}$, the solution u^ν of (5.2), (5.3) belongs to $N^{1,k}_{\mathscr{C}}(\Omega_T)$. Therefore we have already obtained the following result:

Proposition 5.1. *Under the assumption of Theorem* 1.2, *the solution* u^ν *of* (5.2), (5.3) *belongs to* $C([0,T], H^{s+1}(\bar{\Omega}_t)) \cap C^1([0,T], H^s(\bar{\Omega}_t)) \cap \cap N^{1,k}_{\mathscr{C}}(\Omega_T)$ *for any* $T \in (0, T_0]$ *and* $\nu \in \{0, 1, 2, \ldots\}$.

Let us discuss some properties of the solution sequence $\{u^\nu\}_{\nu=0}^\infty$.

Proposition 5.2. *Given* $T_0 > 0$, *there is* $T_1 \in (0, T_0]$ *such that the sequence* $\{uv\}_{v=0}^\infty$ *is bounded in* $C([0, T_1], H^{s+1}(\bar{\Omega}_t)) \cap C^1([0, T_1], H^s(\bar{\Omega}_t)) \cap N^{1,k}_{\mathscr{C}}(\Omega_{T_1})$.

Proof. At first, we establish esimate for $|||u^\nu|||_{s,T}$. Applying Lemma 3.1 to the problem (5.2) implies

$$|||u^{\nu+1}|||_{s,T} \leqslant C_4\Bigg(\|u_0\|_{H^{s+1}(\omega)} + \|u_1\|_{H^s(\omega)} + + \left(\int_0^T \| f(u^\nu, \nabla u^\nu) \|^2_{H^s(\bar{\Omega}_t)} dt \right)^{1/2} \Bigg). \tag{5.5}$$

From our discussion, it is easy to see that the estimate (5.4) also holds if we substitute u^ν for u^0 with any $\nu \in \{1, 2, \ldots\}$. Hence, (5.5) is reduced to

$$|||u^{v+1}|||_{s,T} \leqslant C_4(\|u_0\|_{H^{s+1}(\omega)} + \|u_1\|_{H^s(\omega)} + \sqrt{T}G(|||u^v|||_{s,T})). \quad (5.6)$$

Let $r_1 = C_4(\|u_0\|_{H^{s+1}(\omega)} + \|u_1\|_{H^s(\omega)})$, $r = \max(r_1 + 1, |||u^0|||_{s,T_0})$, $R_1 = \|u_0\|_{N^{1,k}_{\{0\}}(\omega)} + \|u_1\|_{N^k_{\{0\}}(\omega)}$ and $R = \|u^0\|_{N^{1,k}_{\mathscr{C}}(\Omega_{T_0})} + 1$. We want to prove that there is a $T_1 \in (0, T_0]$ such that

$$|||u^v|||_{s,T_1} \leqslant r \quad \text{and} \quad \|u^v\|_{N^{1,k}_{\mathscr{C}}(\Omega_{T_1})} \leqslant R \quad (5.7)$$

for any $v \in \{0, 1, 2, \ldots\}$, which we will prove by induction over v.

From the above choice of r and R, the estimate (5.7) is obviously satisfied by u^0.

Assuming (5.7) valid for the case v, it is sufficient to prove the case $v+1$. Using Proposition 2.1 and Lemma 5.1 we know that the inductive hypothesis implies

$$\|f(u^v, \nabla u^v)\|_{N^k_{\mathscr{C}}(\Omega_T)} \leqslant C(|||u^v|||_{s,T}) \sup_{1 \leqslant k_1 \leqslant k} (|||u^v|||_{s,T} + \|u^v\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)})^{k_1} \leqslant$$
$$\leqslant C(r)(r+R)^{\max\{1,k\}} \quad (5.8)$$

where $C(\cdot)$ is a continuous increasing function, and we always take the value of k_1 as 1 for the case $k = 0$ in the above expression.

Let $T_1 \in (0, T_0]$ satisfy:

$$\begin{cases} C_1\sqrt{T_1}(R_1 + C(r)(r+R)^{\max\{1,k\}}) \leqslant R \\ r_1 + C_4\sqrt{T_1}G(r) \leqslant r \end{cases} \quad (5.9)$$

with C_1 being indicated as in Proposition 3.1.

For the problem (5.2), using (3.3), (5.6) it follows that (5.7) holds for the case $v+1$ from the choice of T_1 in (5.9). ■

Proposition 5.3. *For T_1 given in Proposition 5.2, there is $T_2 \in (0, T_1]$ such that the sequence $\{u^v\}_{v=1}^{\infty}$ is convergent in $C([0, T_2], H^{s+1}(\bar{\Omega}_t)) \cap \cap C^1([0, T_2], H^s(\bar{\Omega}_t)) \cap N^{1,k}_{\mathscr{C}}(\Omega_{T_2})$.*

Proof. For any $v \in \{1, 2, 3, \ldots\}$, $u^{v+1} - u^v$ satisfies

$$\begin{cases} P(u^{v+1} - u^v) = f(u^v, \nabla u^v) - f(u^{v-1}, \nabla u^{v-1}) \\ (u^{v+1} - u^v)|_{t=0} = 0 \\ (u^{v+1} - u^v)_t|_{t=0} = 0. \end{cases} \quad (5.10)$$

We decompose the right hand side of the above equation into

$$f(u^\nu,\nabla u^\nu)-f(u^{\nu-1},\nabla u^{\nu-1})=f(u^\nu,\nabla u^\nu)-f(u^{\nu-1},\nabla u^\nu)+ \\ +f(u^{\nu-1},\nabla u^\nu)-f(u^{\nu-1},\nabla u^{\nu-1}) \tag{5.11}$$

and we have

$$\|f(u^\nu,\nabla u^\nu)-f(u^{\nu-1},\nabla u^\nu)\|_{N^k_{\mathscr{C}}(\Omega_T)}\leqslant \\ \leqslant C_1(\|u^\nu-u^{\nu-1}\|_{L^\infty(\Omega_T)}+\|u^\nu-u^{\nu-1}\|_{N^k_{\mathscr{C}}(\Omega_T)}) \tag{5.12}$$

by using the usual mean value theorem and Propositions 2.1, 5.2. On the other hand, since $s>\frac{n}{2}$, using Lemma 5.1 it follows that $\|u^\nu-u^{\nu-1}\|_{L^\infty(\Omega_T)}$ can be controlled by $C|||u^\nu-u^{\nu-1}|||_{s,T}$ which is substituted into (5.12) implies

$$\|f(u^\nu,\nabla u^\nu)-f(u^{\nu-1},\nabla u^\nu)\|_{N^k_{\mathscr{C}}(\Omega_T)}\leqslant \\ \leqslant C_2(|||u^\nu-u^{\nu-1}|||_{s,T}+\|u^\nu-u^{\nu-1}\|_{N^k_{\mathscr{C}}(\Omega_T)}). \tag{5.13}$$

Similarly, we can establish

$$\|f(u^{\nu-1},\nabla u^\nu)-f(u^{\nu-1},\nabla u^{\nu-1})\|_{N^k_{\mathscr{C}}(\Omega_T)}\leqslant \\ \leqslant C_3(\|u^\nu-u^{\nu-1}\|_{W^{1,\infty}(\Omega_T)}+\|u^\nu-u^{\nu-1}\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)})\leqslant \\ \leqslant C_4(|||u^\nu-u^{\nu-1}|||_{s,T}+\|u^\nu-u^{\nu-1}\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)}). \tag{5.14}$$

Substituting (5.13), (5.14) into (5.11) it follows

$$\|f(u^\nu,\nabla u^\nu)-f(u^{\nu-1},\nabla u^{\nu-1})\|_{N^k_{\mathscr{C}}(\Omega_T)}\leqslant \\ \leqslant C_5(|||u^\nu-u^{\nu-1}|||_{s,T}+\|u^\nu-u^{\nu-1}\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)}). \tag{5.15}$$

Using Proposition 3.1 and (5.15) for the problem (5.10) we obtain

$$\|u^{\nu+1}-u^\nu\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)}\leqslant \\ \leqslant C_6\sqrt{T}(|||u^\nu-u^{\nu-1}|||_{s,T}+\|u^\nu-u^{\nu-1}\|_{N^{1,k}_{\mathscr{C}}(\Omega_T)}) \tag{5.16}$$

for any $T\in(0,T_1]$.

Applying the estimate (5.6) to (5.10), and using the mean value theorem and Schauder's Lemma (see Lemma 1.3 of [3]) it follows

$$|||u^{\nu+1}-u^\nu|||_{s,T}\leqslant C\sqrt{T}|||u^\nu-u^{\nu-1}|||_{s,T} \tag{5.17}$$

being valid.

Combining (5.16) with (5.17) yields our conclusion of this proposition. ■

Summarizing Propositions 5.2 and 5.3 immediately produces.

Theorem 5.1. *Under the condition of Theorem* 1.2, *there is* $T_* > 0$ *such that the problem* (1.3) *admits a unique solution u in* $C([0, T_*], H^{s+1}(\bar{\Omega}_t)) \cap C^1([0, T_*], H^s(\bar{\Omega}_t)) \cap N_{\mathscr{C}}^{1,k}(\Omega_{T_*})$.

Obviously, the above theorem directly implies the conclusion of Theorem 1.2 by using the usual Sobolev embedding theorem.

Remark 5.1. By using Lemma 3.2, we know that the solution u^ν of (5.2) and (5.3) belongs to $H^{s+1}(\Omega_T)$ for any $\nu \in \{0, 1, 2, \ldots\}$ and $T \in [0, T_0]$. Since $u^\nu \in W^{1,\infty}(\Omega_T)$ obtained before Proposition 5.1, and $H^s \cap L^\infty(\Omega_T)$ is an algebra, we have that the solution u of (1.3) constructed above also belongs to $H^{s+1}(\Omega_{T_*})$ while $T_* > 0$ small enough by using uniqueness of (1.3).

For the case of the nonlinear term $f(t, x, u, \nabla u)$ of (1.3) being independent of ∇u, Remark 1.2 can be discussed by the same argument as above, and its proof is simpler than Theorem 1.2 in notations.

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